

Yang-Mills classical solutions and fermionic zero modes from lattice calculations *

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We study a series of problems in classical Yang-Mills theories using lattice methods. We first investigate $SU(N)$ self-dual configurations on the torus with twisted boundary conditions. We also study the zero modes of the Dirac equation in topological non-trivial background Yang-Mills fields.

1. Introduction

It has long been argued that classical configurations might play a relevant role in the understanding of low-energy phenomena like confinement and chiral symmetry breaking. Here we will present a study of a particular kind of solutions which appear on the torus when twisted boundary conditions are imposed. These are instanton-like configurations with the peculiarity of having fractional topological charge. In the large N limit their action scales as $8\pi^2/Ng^2$. As has been argued in [1] they may be important to understand the long-distance dynamics of QCD. In this paper we will present a preliminary investigation of: i) the large N limit behaviour ii) the zero modes of the Dirac operator on the background of fractional-charge instantons.

2. Yang-Mills classical solutions

The first part of this work is a study of $SU(N)$ (anti) self-dual Yang-Mills classical solutions on the torus with twisted boundary conditions. It is known that in this case the topological charge Q is not always an integer [2], but is given by

$$Q = \frac{1}{16\pi^2} \int Tr \left(F_{\mu\nu} \tilde{F}_{\mu\nu} \right) d_4x = \nu - \frac{\kappa}{N}, \quad (1)$$

where ν and κ are integers, $F_{\mu\nu}$ is the Yang-Mills strength tensor in the fundamental representation

and $\tilde{F}_{\mu\nu}$ its dual. Here

$$\kappa = \frac{1}{4} n_{\mu\nu} \tilde{n}_{\mu\nu} = \vec{k} \cdot \vec{m}, \quad (2)$$

with $n_{\mu\nu}$ the usual twist tensor and $k_i = n_{0i}$, $n_{ij} = \epsilon_{ijk} m_k$.

From Schwarz-inequality

$$S = \frac{1}{2} \int Tr (F_{\mu\nu} F_{\mu\nu}) d_4x \geq 8\pi^2 |Q| \quad (3)$$

one easily derives that if $\kappa \neq 0$ (modulo N) an obstruction is found for zero-action configurations. We are interested in those solutions with minimal non-trivial action

$$S = 8\pi^2 |Q| = \frac{8\pi^2}{N}, \quad (4)$$

on a volume $[0, L]^3 \times [0, T]$, with $T \gg L$. Some of them are already known: i) For specific values of N , 't Hooft has constructed non-abelian solutions with constant field strength which turn out to be (anti)self-dual whenever the sides of the torus satisfy certain relations (see [3] for details), ii) there are also numerical studies of the solution for $SU(2)$ with twist $\vec{m} = (1, 1, 1)$ and $\vec{k} = (1, 1, 1)$ [4]. The present work is inspired by this last reference.

We have restricted our analysis to the following twist tensors:

1. Spatial twist, always $\vec{m} = (1, 1, 1)$.
2. Temporal twist, two cases:

*Presented by A. Montero at Lattice '97

- $\vec{k} = (1, 0, 0)$ for $N = 3, 4, \dots, 13$ the solution is in this case antiself-dual, $Q = -1/N$.
- $\vec{k} = (n, n, n)$ for $N = 3n + 1 = 4, 7, 10, 13$ the solution is here self-dual, $Q = 1/N$.

The solutions were generated on $N_s^3 \times N_t$ lattices ($N_t \gg N_s$) by cooling with the Cabibbo-Marinari-Okawa algorithm [5]. They verify the following properties:

1. Scaling towards the continuum solution.
2. Self-duality or antiself-duality.
3. The energy profile

$$\epsilon(t) = \int S(\vec{x}, t) d_3 \vec{x} \quad (5)$$

with $S(\vec{x}, t)$ the action density, is instantonic and independent of temporal twist.

4. Behavior for large N:

- The energy profile behaves as $\epsilon(t) \sim f(t/N)/N^2$ as illustrated in Figure 1.
- The norm of the field strength tensor becomes independent of the spatial coordinates.
- In the gauge

$$A_0 = 0 \quad A_i(t = -\infty, \vec{x}) = 0, \quad (6)$$

one eigenvalue (the same one for A_i, B_i, E_i) dominates the solution in the sense that it gives the major contribution to the action.

A more detailed description of the properties of these solutions will be presented in [6].

3. Fermionic zero modes

The second part of this work is a study of fermionic zero modes of the Dirac equation in the background of topologically non-trivial Yang-Mills fields.

To obtain the zero modes we have used the lattice approach with Wilson fermions and also with

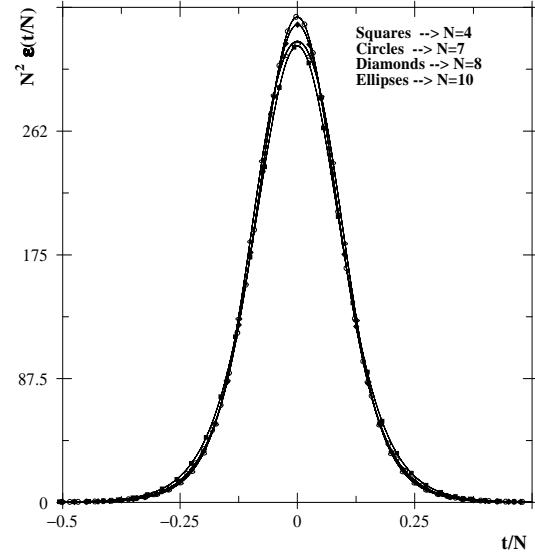


Figure 1. The energy profile for 4 different $SU(N)$ solutions multiplied by N^2 is plotted as a function of t/N .

naive (Kogut-Susskind) fermions. In the massless case, the Dirac operator reads

$$\mathcal{D}\Psi(n) = -\frac{1}{2} \sum_{\mu} [(r - \gamma_{\mu}) U_{\mu}(n) \Psi(n + \hat{\mu}) + (r + \gamma_{\mu}) U_{\mu}^{\dagger}(n - \hat{\mu}) \Psi(n - \hat{\mu})] + 4r\Psi(n),$$

where Ψ is the fermion field, U_{μ} the gauge field, γ_{μ} the Dirac matrices and r the Wilson parameter.

We look for the lowest eigenvalues (and the corresponding eigenvectors) of the operator $(\gamma_5 \mathcal{D})^2$ using three different methods:

- Conjugate gradient
- Local minimization
- Lanczos (only for eigenvalues)

The results we have obtained are the same independently of the method employed in the diagonalization. We should point out that the Wilson parameter must be big enough to avoid fermionic doubling and, at the same time, small enough such as not to distort the solution. For a lattice size N_s^4 with $N_s \geq 8$ a good value is $r = 0.001$.

Table 1

The three lowest eigenvalues of $(\gamma_5 \not{D})^2$ in the presence of a $Q = 1/2$ instanton on a N_s^4 lattice.

N_s	$1^{st} (\times 10^{-9})$	$2^{nd} (\times 10^{-9})$	$3^{rd} (\times 10^{-6})$
11	6.946	6.946	4.150
12	4.835	4.835	4.140
13	3.546	3.546	4.107
14	2.633	2.633	4.103

As a check of the method we successfully extracted the zero mode for the instanton solution, also known as the 't Hooft zero mode.

Here, we will present our results for the zero modes in the background of the $SU(2)$ self-dual gauge field with $Q = 1/2$ on the $[0, L]^4$ torus with twist $\vec{m} = \vec{k} = (1, 1, 1)$. Fermions are taken in the adjoint representation of $SU(2)$ (to avoid twist singularities).

We have worked on N_s^4 lattices with $N_s = 11, 12, 13, 14$ and $r = 0.001$. The three lowest eigenvalues obtained are shown in Table 1. We see that two of these eigenvalues scale to zero. These would correspond to the two zero modes predicted by the Index Theorem. They are related to each other by the operation $\Psi_2 = \gamma_1 \gamma_3 \Psi_1^*$. Our results show that the gauge invariant density of these lowest eigenvectors scales nicely towards a continuum curve. Furthermore, this density is cubic symmetric and peaks at the position of the $Q = 1/2$ instanton. This is exemplified by Figure 2, where we display the quantity:

$$\Phi(x_4) = \sum_{\alpha i} \int \Psi_{\alpha i}^* \Psi_{\alpha i} dx_1 dx_2 dx_3. \quad (7)$$

In the above equation, x_4 is one of $\{x, y, z, t\}$, and x_1, x_2, x_3 are the other three components, where the origin of coordinates is taken at the maximum of the gauge field configuration. It is clear from the figure the nice scaling of the results. The curves for naive fermions are not displayed, but fall on top of the other.

A more detailed description of these solutions and of others obtained for different classical Yang-Mills backgrounds will be given in [7].

Acknowledgments: DFG financial support

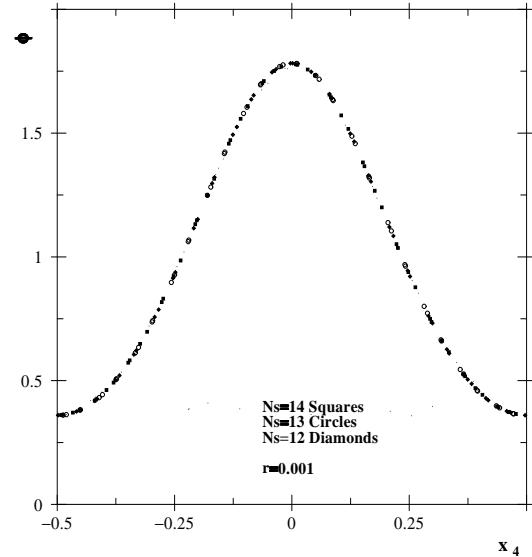


Figure 2. The quantity Φ is plotted as a function of x_4 for different lattice sizes.

for MGP is thankfully acknowledged. AGA and AM would like to thank CICYT (Grant AEN 96 – 1664) for their financial support.

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